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Bessel Wavelets and the Galerkin Analysis of the Bessel Operator

Michael Frazier

E-mail: frazier@math.msu.edu

and

Shangqian Zhang

Department of Advanced Engineering, Delphi Packard Electric Systems, MS 47C, P.O. Box 431, Warren, Ohio 44486
E-mail: shangqian.zhang@eng.ped.gmeds.com

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We construct a complete orthonormal system $\{\psi_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{N}}$ for $L^2(\mathbb{R}_+, dx)$ such that each *Bessel wavelet* $\psi_{j,k}$ has a compactly supported Hankel transform. We apply this system in the Galerkin solution of the differential equation $Lu = f$ on $\mathbb{R}_+$, where L is the Bessel operator. The associated linear system can be preconditioned using a simple diagonal preconditioning matrix to give a system which is sparse and has condition number which is bounded independent of the Galerkin subspace. Thus we obtain the same advantages for the singular operator L that have been obtained for uniformly elliptic operators using standard wavelets. The main step is a characterization of the Bessel–Sobolev norm of a function in terms of its Bessel wavelet coefficients. We also present numerical work which shows that the condition numbers are small enough for this method to be practical.

Key Words: wavelets; Galerkin procedure; Bessel function; numerical solution of Bessel's equation; Hankel transform.

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1. INTRODUCTION

Let $\nu > 0$ (the case $\nu = 0$ will be considered in Section 7). Let $\mathbb{R}_+$ be the open half-line $(0, +\infty)$ and let $AC_{loc}(\mathbb{R}_+)$ be the set of complex-valued functions on $\mathbb{R}_+$ that are locally absolutely continuous. For $g \in AC_{loc}(\mathbb{R}_+)$ such that $g' \in AC_{loc}(\mathbb{R}_+)$, define

$$\tilde{L}g(x) = -g''(x) + \frac{\nu^2 - 1/4}{x^2} g(x), \tag{1}$$

which exists pointwise a.e. on $\mathbb{R}_+$. To define a self-adjoint operator corresponding to $\tilde{L}$, first let $\tilde{L}|_{C_0^\infty}$ be the restriction of $\tilde{L}$ to $C_0^\infty(\mathbb{R}_+)$, the C^∞ functions with compact support in $\mathbb{R}_+$. Let $L = L_\nu$ denote the Friedrichs (Dirichlet) extension of $\tilde{L}|_{C_0^\infty}$ in $L^2 = L^2(\mathbb{R}_+, dx)$ (see, e.g., [10, p. 325]). The domain of L is

$$\mathscr{D}_L = \left\{ g \colon g, g' \in AC_{loc}(\mathbb{R}_+);\ \lim_{x \to 0+} g(x) = 0;\ g, g', \tilde{L}g \in L^2 \right\} \tag{2}$$

([9, p. 234]). Then L is self-adjoint and $Lg = \tilde{L}g$ for $g \in \mathscr{D}_L$. We call L the *Bessel operator* because of its connections to Bessel functions described below. Our purpose is to construct *Bessel wavelets* and apply these in the Galerkin approach to the numerical solution of the equation

$$Lu = f, \tag{3}$$

for $f \in \mathscr{R}(L)$, the range of L. Our goal is to obtain the same advantages for the singular operator L as obtained with standard wavelets in the case of a uniformly elliptic Sturm–Liouville operator (e.g., in [8]). Specifically, the associated linear system can be preconditioned via a simple diagonal preconditioner to have a bounded condition number while still being sparse.

The Bessel function of order ν is defined for $x > 0$ by

$$J_\nu(x) = \alpha_\nu x^\nu \int_{-1}^{+1} (1 - t^2)^{\nu - 1/2} e^{ixt}\, dt, \tag{4}$$

where $\alpha_\nu = (\pi^{1/2} 2^\nu \Gamma(\nu + 1/2))^{-1}$ ([6, p. 138]; this formula is valid for $\nu > -1/2$). Alternately,

$$J_\nu(x) = x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + \nu + 1)} \left(\frac{x}{2} \right)^{2k} \tag{5}$$

([6, p. 130]; this formula is valid whenever ν is not a negative integer, while $J_{-n} = (-1)^n J_n$ for $n \in \mathbb{N}$). By (4), $\tilde{L}(\sqrt{x} J_\nu(x)) = \sqrt{x} J_\nu(x)$. Consequently

$$\tilde{L}_{(x)}\big(\sqrt{x} J_\nu(x\xi)\big) = \xi^2 \sqrt{x} J_\nu(x\xi) \tag{6}$$

for each $\xi > 0$, where the subscript in $\tilde{L}_{(x)}$ indicates that $\tilde{L}$ is applied in the x variable. Thus $\sqrt{x} J_\nu(x\xi)$ plays the same role with respect to $\tilde{L}$ that the characters $\{e^{ix\xi}\}_{\xi \in \mathbb{R}}$ play with respect to $-d^2/dx^2$. We define a Bessel analog of the Fourier transform by

$$\hat{f}(\xi) = \lim_{R \to +\infty} \int_0^R f(x) \sqrt{x} J_\nu(x\xi)\, dx \tag{7}$$

for $\xi > 0$, where the limit exists in $L^2(\mathbb{R}_+, \xi\, d\xi)$ for every $f \in L^2(\mathbb{R}_+, dx)$. Moreover,

$$\hat{}: L^2(\mathbb{R}_+, dx) \to L^2(\mathbb{R}_+, \xi\, d\xi) \tag{8}$$

is unitary, with inverse $^\vee$ defined by

$$g^\vee(x) = \lim_{R \to +\infty} \int_0^R g(\xi) \sqrt{x} J_\nu(x\xi)\, \xi\, d\xi. \tag{9}$$

These facts follow from the corresponding facts ([4, p. 1535] or [1, p. 213]) for the Hankel transform H, which satisfies $Hf(\xi) = \xi^{1/2}\hat{f}(\xi)$ by definition.

We use the notations $\langle f, g\rangle$ for the usual inner product $\int_0^\infty f(x)\overline{g(x)}\, dx$ in $L^2(\mathbb{R}_+, dx)$ and $\langle f, g\rangle_\xi$ for the weighted inner product $\int_0^\infty f(\xi)\overline{g(\xi)}\,\xi\, d\xi$ in $L^2(\mathbb{R}_+, \xi\, d\xi)$. The unitarity of $\hat{}$ means that the Parseval relation

$$\langle \hat{f}, \hat{g}\rangle_\xi = \langle f, g\rangle \tag{10}$$

holds. For all $g \in \mathscr{D}_L$,

$$(Lg)^\wedge(\xi) = \xi^2 \hat{g}(\xi). \tag{11}$$

For $g \in C_0^\infty(\mathbb{R}_+)$, (11) follows from (6) and integration by parts (the formal self-adjointness of $\tilde{L}$). For general $g \in \mathscr{D}_L$, it follows from the spectral theorem applied to the differential operator L ([4, p. 1535] or [1, p. 192]).

The Bessel analogs of the homogeneous Sobolev spaces, which we denote H_B^s, are defined for $s \in \mathbb{R}$ by

$$H_B^s = \big\{ f \in L^2(\mathbb{R}_+) : \|f\|_{H_B^s} < +\infty \big\},$$

where

$$\|f\|_{H_B^s} = \left(\int_0^\infty \xi^{2s} \left| \hat{f}(\xi) \right|^2 \xi \, d\xi \right)^{1/2}. \tag{12}$$

The requirement that $f \in L^2$ is made because we have only defined $\hat{}$ on L^2, but note that the norm is homogeneous. By the unitarity of $\hat{}$, $H_B^0 = L^2(\mathbb{R}_+, dx)$, with the same norm. For $g \in \mathscr{D}_L$, (10)–(11) imply

$$\|g\|_{H_B^1}^2 = \left\langle \xi^2 \hat{g}(\xi), \hat{g}(\xi) \right\rangle_\xi = \left\langle (Lg)\hat{}, \hat{g} \right\rangle_\xi = \langle Lg, g \rangle. \tag{13}$$

To describe our purpose, we summarize some of Jaffard's work in [8]. Consider a Sturm–Liouville problem on $[0, 1]$ of the form

$$-\frac{d}{dx}\left(a(x) \frac{du}{dx} \right) + b(x)u(x) = f(x), \tag{14}$$

with $u(0) = u(1) = 0$, assuming the uniform ellipticity conditions

$$0 < c_1 \le a(x) \le c_2 < +\infty, \qquad 0 \le b(x) \le c_3 < +\infty \tag{15}$$

for all $x \in [0, 1]$. (Jaffard considers the multidimensional case, but we only discuss the one-dimensional case here.) Using wavelets adapted to $[0, 1]$ in a Galerkin approximation to the solution of (14), one obtains a linear system of equations. Jaffard shows that this system can be preconditioned using a diagonal preconditioning matrix in such a way that the matrix M of the resulting linear system $Mx = y$ has two key properties: M is sparse and the condition number of M is bounded independent of the degree of approximation (i.e., the size of the subspace used in the finite element process). Recall that the condition number of a matrix M is defined by $\|M\| \, \|M^{-1}\|$, and the smaller the condition number of M, the more stable the solution of the system $Mx = y$ is under perturbation of the data y. Our goal is to obtain the advantages of sparseness and the bounded condition number in the case of the Bessel operator, which is obviously not uniformly elliptic. These results are stated in Theorem 10 and Lemma 12. Our approach is to construct wavelets which are adapted to L in the same way that standard wavelets are adapted to $-d^2/dx^2$.

More precisely, using Meyer's original wavelets ([11]), we will construct a family

$$\{\psi_{j,k}\}_{j \in \mathbb{Z},\, k \in \mathbb{N}},$$

which forms an orthonormal basis for $L^2(\mathbb{R}_+)$ (Theorem 3). These *Bessel wavelets* satisfy

$$\operatorname{supp} \hat{\psi}_{j,k} \subseteq \left\{ \xi : \frac{2\pi}{3} 2^j \le \xi \le \frac{8\pi}{3} 2^j \right\}. \tag{16}$$

They are also localized in space, near the point $2^{-j}k$ (Theorem 6): there exists $c_M < \infty$ such that

$$\left|\psi_{j,k}(x)\right| \le c_M 2^{j/2}\left(1 + 2^j x/k\right)^{-M}\left(1 + 2^j|x - 2^{-j}k|\right)^{-M} \times \min\left(\left(2^j x\right)^{\nu+1/2}, 1\right), \tag{17}$$

where M is determined by the smoothness of a function w chosen in the construction (if w is chosen to be C^∞, then (17) holds for all $M > 0$). The Bessel wavelets do not have the translation structure of standard wavelets, but they do have the following dilation property:

$$\psi_{j,k}(x) = 2^{j/2}\psi_{0,k}(2^j x). \tag{18}$$

The key fact leading to the control on the condition number in Jaffard's work is the well-known characterization of the H^1 norm in terms of wavelet coefficients. In our case the corresponding result (Theorem 8) is

$$\|f\|^2_{H^1_B} \approx \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}}\left|2^j\langle f, \psi_{j,k}\rangle\right|^2, \tag{19}$$

where $\approx$ is standard notation indicating that there exist positive constants, independent of f, bounding the ratio of the two quantities above and below. With this result, we can show that using the family $\{\psi_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{N}}$ in the Galerkin procedure to solve (3) leads to a linear system that can be diagonally preconditioned to a sparse system with bounded condition number.

Suppose T has the form

$$T = -\frac{d}{dx}\left(a(x)\frac{d}{dx}\right) + \frac{b(x)}{x^2}, \tag{20}$$

where we assume there exist constants c_1, c_2 such that $0 < c_1 \le a(x), b(x) \le c_2 < +\infty$ for all $x > 0$. Suppose $\nu > 1/2$, and consider the Galerkin solution of $Tu = f$ using Bessel wavelets for L_ν. Then we still obtain the boundedness of the condition number of the associated preconditioned linear system (Lemma 11). Thus Bessel wavelets can be used to treat perturbations of the Bessel operator that have singularities of the same order.

This paper is organized as follows. In Section 2, we develop a Bessel version of the φ-transform (see [7]). This has the advantage that the analog of (19) is easy to prove. In the next section we construct Bessel wavelets and prove that they form an orthonormal basis for $L^2(\mathbb{R}_+, dx)$. We also demonstrate (17) and show that the Bessel wavelets belong to $\mathscr{D}_L$. By applying Schur's lemma and the results of Section 2, the norm equivalence (19) is proved in Section 4. The results regarding the Galerkin procedure are proved in Section 5. In Section 6, we present some numerical results. We discuss the case $\nu = 0$ in Section 7. It is distinct because for this case we do not obtain that the Bessel wavelets belong to $\mathscr{D}_L$. However, they do belong to the domain of the maximal extension $L_{\max}$, and we obtain the same numerical results as in the case $\nu > 0$, although by slightly different arguments. In Section 8 we make some concluding comments.

2. THE BESSEL φ-TRANSFORM

Let γ be a C^∞, real-valued function on $(0, \infty)$ satisfying

$$\operatorname{supp} \gamma \subseteq [1/4, 1] \tag{21}$$

and

$$\sum_{j \in \mathbb{Z}} \left(\gamma(2^j \xi)\right)^2 = 1 \qquad \text{for } \xi > 0. \tag{22}$$

For $j \in \mathbb{Z}$, $k \in \mathbb{N}$, and $x > 0$ define

$$\varphi_{j,k}(x) = 2^{-(j-1)/2} \sqrt{x} \int_0^\infty \sin(2^{-j} k \pi \xi) \gamma(2^{-j} \xi) J_\nu(x\xi) \sqrt{\xi}\, d\xi. \tag{23}$$

Note that, by (9),

$$\hat{\varphi}_{j,k}(\xi) = 2^{-(j-1)/2} \xi^{-1/2} \sin(2^{-j} k \pi \xi) \gamma(2^{-j} \xi). \tag{24}$$

THEOREM 1. *Suppose $f \in L^2(\mathbb{R}_+, dx)$. Then*

$$f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}} \langle f, \varphi_{j,k} \rangle \varphi_{j,k} \tag{25}$$

and

$$\|f\|^2_{L^2(\mathbb{R}_+, dx)} = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}} \left|\langle f, \varphi_{j,k} \rangle\right|^2. \tag{26}$$

The precise meaning of (25) *is that the partial sums*

$$f_N = \sum_{j=-N}^{+N} \sum_{k \in \mathbb{N}} \langle f, \varphi_{j,k} \rangle \varphi_{j,k}$$

converge to f in the $L^2(\mathbb{R}_+, dx)$ norm as $N \to \infty$.

Proof. Observe that, for $j \in \mathbb{Z}$,

$$\left\{2^{-(j-1)/2} \sin(2^{-j} k\pi x)\right\}_{k=1}^{\infty} \tag{27}$$

is an orthonormal basis for $L^2([0, 2^j], dx)$. Hence, using (24) and (10), we have that, for $\xi \in [0, 2^j]$,

$$\begin{aligned}
&\gamma(2^{-j}\xi)\sqrt{\xi}\hat{f}(\xi) \\
&\quad = \sum_{k=1}^{\infty} \int_0^{2^j} \gamma(2^{-j}t)\sqrt{t}\hat{f}(t) 2^{-(j-1)/2} \sin(2^{-j}k\pi t)\, dt\, 2^{-(j-1)/2} \\
&\qquad \times \sin(2^{-j}k\pi\xi) \\
&\quad = \sum_{k=1}^{\infty} \langle \hat{f}, \hat{\varphi}_{j,k} \rangle_{\xi} 2^{-(j-1)/2} \sin(2^{-j}k\pi\xi) \\
&\quad = 2^{-(j-1)/2} \sum_{k=1}^{\infty} \langle f, \varphi_{j,k} \rangle \sin(2^{-j}k\pi\xi).
\end{aligned}$$

Therefore, noting by (21) that $\gamma(2^{-j}\xi)$ is supported in $[2^{j-2}, 2^j]$,

$$\begin{aligned}
&\sqrt{x} \int_{\mathbb{R}} \left(\gamma(2^{-j}\xi)\right)^2 \hat{f}(\xi) J_\nu(x\xi)\, \xi\, d\xi \\
&\quad = \sum_{k=1}^{\infty} \langle f, \varphi_{j,k} \rangle \sqrt{x} \int_0^{\infty} 2^{-(j-1)/2} \\
&\qquad \times \sin(2^{-j}k\pi\xi)\gamma(2^{-j}\xi)\sqrt{\xi} J_\nu(x\xi)\, d\xi \\
&\quad = \sum_{k=1}^{\infty} \langle f, \varphi_{j,k} \rangle \varphi_{j,k}(x).
\end{aligned}$$

Taking the Hankel transform of this relation and using (9), we have

$$\hat{f}_N(\xi) = \sum_{j=-N}^{N} \left(\gamma(2^{-j}\xi)\right)^2 \hat{f}(\xi).$$

Therefore

$$\begin{aligned}
\|f - f_N\|^2_{L^2(dx)} &= \|\hat{f} - \hat{f}_N\|^2_{L^2(\xi\, d\xi)} \\
&= \int_0^{\infty} \left|\hat{f}(\xi)\right|^2 \left(1 - \sum_{j=-N}^{N} \left(\gamma(2^{-j}\xi)\right)^2\right) \xi\, d\xi,
\end{aligned}$$

which goes to 0 as $N \to \infty$ by (22) and the dominated convergence theorem. This establishes (25). To obtain (26), substitute (25) for one of the f's in $\langle f, f \rangle$. ∎

The collection $\{\varphi_{j,k}\}_{j \in \mathbb{Z}, k \in \mathbb{N}}$ is not an orthonormal basis for $L^2(\mathbb{R}_+, dx)$, but by (26) it is a tight frame. We call the map taking f to $\{\langle f, \varphi_{j,k} \rangle\}_{j \in \mathbb{Z}, k \in \mathbb{N}}$ the *Bessel φ-transform*. It is easy to characterize the H_B^s norm in terms of the Bessel φ-transform.

THEOREM 2. *Let $s \in \mathbb{R}$. For $f \in L^2(\mathbb{R}_+)$,*

$$\|f\|_{H_B^s} \approx \left(\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}} \left| 2^{js} \langle f, \varphi_{j,k} \rangle \right|^2 \right)^{1/2}. \tag{28}$$

Proof. Applying (22) and the support properties of $\gamma(2^{-j}\xi)$,

$$\begin{aligned} \|f\|_{H_B^s}^2 &= \sum_{j \in \mathbb{Z}} \int_{2^{j-2}}^{2^j} \left(\gamma(2^{-j}\xi)\right)^2 \xi^{2s} \left|\hat{f}(\xi)\right|^2 \xi \, d\xi \\ &\approx \sum_{j \in \mathbb{Z}} 2^{2js} \int_{2^{j-2}}^{2^j} \left(\gamma(2^{-j}\xi)\right)^2 \left|\hat{f}(\xi)\right|^2 \xi \, d\xi. \end{aligned}$$

By the support condition $\gamma(2^{-j}\xi)$ and the fact that the set in (27) is an orthonormal basis for $L^2([0, 2^j], dx)$,

$$\begin{aligned} &\int_{2^{j-2}}^{2^j} \left(\gamma(2^{-j}\xi)\right)^2 \left|\hat{f}(\xi)\right|^2 \xi \, d\xi \\ &\quad = \sum_{k=1}^{\infty} \left| \int_0^{2^j} \gamma(2^{-j}\xi) \hat{f}(\xi) \sqrt{\xi}\, 2^{-(j-1)/2} \sin\left(2^{-j} k \pi \xi\right) d\xi \right|^2 \\ &\quad = \sum_{k=1}^{\infty} \left| \langle \hat{f}, \hat{\varphi}_{j,k} \rangle_\xi \right|^2 = \sum_{k=1}^{\infty} \left| \langle f, \varphi_{j,k} \rangle \right|^2, \end{aligned}$$

using (24) and (10). Substituting this above completes the proof. ∎

Observe that (28) is an equality when $s = 0$, by (26).

3. BESSEL WAVELETS

In this section, we will use Meyer's original wavelets ([11]) to construct a certain orthonormal basis for $L^2(\mathbb{R}_+, dx)$. To describe Meyer's wavelets, we follow Daubechies' exposition ([3]). We define the Fourier transform $\mathscr{F}$ by

$$\mathscr{F} f(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx.$$

With this normalization, the inverse Fourier transform is defined by $\mathscr{F}^{-1}f(x) = \mathscr{F}f(-x)$, and we have the Parseval relation

$$\langle \mathscr{F}f, \mathscr{F}g \rangle = \langle f, g \rangle. \tag{29}$$

Meyer constructed a function, which we call h, such that the collection $\{h_{j,k}\}_{j,k\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\mathbb{R})$ $(= L^2(\mathbb{R}, dx))$, where $h_{j,k}(x) = 2^{j/2}h(2^j x - k)$ for each $j, k \in \mathbb{Z}$. To construct h, let $w\colon \mathbb{R} \to \mathbb{R}$ be an even function which is sufficiently smooth, at least C^2 (w can be chosen to be C^∞), and which has compact support as follows:

$$\operatorname{supp} w \subseteq \{\xi \in \mathbb{R}: 2\pi/3 \le |\xi| \le 8\pi/3\}. \tag{30}$$

An example given in [3, p. 117] is

$$w(\xi) = \begin{cases} \sin\left(\dfrac{\pi}{2}\rho\left(\dfrac{3}{2\pi}|\xi| - 1\right)\right), & \dfrac{2\pi}{3} \le |\xi| \le \dfrac{4\pi}{3} \\ \cos\left(\dfrac{\pi}{2}\rho\left(\dfrac{3}{4\pi}|\xi| - 1\right)\right), & \dfrac{4\pi}{3} \le |\xi| \le \dfrac{8\pi}{3} \\ 0, & \text{otherwise}, \end{cases} \tag{31}$$

where ρ is a sufficiently smooth function (at least C^2) satisfying $\rho(x) = 0$ if $x \le 0$, $\rho(x) = 1$ if $x \ge 1$, and $\rho(x) + \rho(1-x) = 1$. Then

$$h = \mathscr{F}^{-1}\big((2\pi)^{-1/2}e^{i\xi/2}w(\xi)\big).$$

It follows that

$$\mathscr{F}h_{j,k}(\xi) = (2\pi)^{-1/2}2^{-j/2}e^{-i2^{-j}(k-1/2)\xi}w(2^{-j}\xi). \tag{32}$$

By (29), the collection

$$\{\mathscr{F}h_{j,k}\}_{j,k\in\mathbb{Z}} \tag{33}$$

is an orthonormal basis for $L^2(\mathbb{R})$.

Our *Bessel wavelets* are defined for $j \in \mathbb{Z}$ and $k \in \mathbb{N}$ by

$$\psi_{j,k}(x) = (2\pi)^{-1/2}2^{-j/2+1}\sqrt{x} \times \int_0^\infty \sin(2^{-j}(k-1/2)\xi)w(2^{-j}\xi)J_\nu(x\xi)\sqrt{\xi}\,d\xi. \tag{34}$$

Note that each $\psi_{j,k}$ is real-valued. Our goal is to prove the following.

THEOREM 3. *The set $\{\psi_{j,k}\}_{j\in\mathbb{Z},\ k\in\mathbb{N}}$ is an orthonormal basis for $L^2(\mathbb{R}_+, dx)$.*

The proof involves two simple steps. First we reflect the functions $\mathscr{F}h_{j,k}$ (Lemma 4), to adapt to $\mathbb{R}_+$, in the same way that the Fourier sine expansion on $\mathbb{R}_+$ is obtained from the usual complex exponential Fourier expansion on $\mathbb{R}$. Then we renormalize to deal with the weight ξ on the transform side and apply (10).

To begin the first step, define, for $j \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$g_{j,k}(\xi) = (2\pi)^{-1/2} 2^{-j/2+1} \sin\left(2^{-j}(k - 1/2)\xi\right) w\left(2^{-j}\xi\right).$$

LEMMA 4. *The collection* $\{g_{j,k}\}_{j \in \mathbb{Z}, k \in \mathbb{N}}$ *is an orthonormal basis for* $L^2(\mathbb{R}_+, dx)$.

Proof. Observe that because w is even,

$$g_{j,k}(\xi) = i\left(\mathscr{F}h_{j,k}(\xi) - \mathscr{F}h_{j,k}(-\xi)\right) = i\left(\mathscr{F}h_{j,k}(\xi) - \mathscr{F}h_{j,-k+1}(\xi)\right).$$

Hence, using the fact that each $g_{j,k}$ is odd,

$$\begin{aligned}
\langle g_{j,k}, g_{j',k'} \rangle &= \int_0^\infty g_{j,k}(x)\overline{g_{j',k'}(x)}\,dx = \tfrac{1}{2}\int_{-\infty}^\infty g_{j,k}(x)\overline{g_{j',k'}(x)}\,dx \\
&= \tfrac{1}{2}\int_{\mathbb{R}} \mathscr{F}h_{j,k}\overline{\mathscr{F}h_{j',k'}} - \tfrac{1}{2}\int_{\mathbb{R}} \mathscr{F}h_{j,-k+1}\overline{\mathscr{F}h_{j',k'}} \\
&\quad - \tfrac{1}{2}\int_{\mathbb{R}} \mathscr{F}h_{j,k}\overline{\mathscr{F}h_{j',-k'+1}} + \tfrac{1}{2}\int_{\mathbb{R}} \mathscr{F}h_{j,-k+1}\overline{\mathscr{F}h_{j',-k'+1}}.
\end{aligned}$$

Note that $k' \neq -k + 1$, since $k, k' \geq 1$, so the two middle integrals are 0. By the orthonormality of the $\mathscr{F}h_{j,k}$'s, the other two integrals are 1 when $(j,k) = (j',k')$ and 0 otherwise. This proves the orthonormality of the set $\{g_{j,k}\}_{j \in \mathbb{Z}, k \in \mathbb{N}}$.

To prove completeness, suppose $f \in L^2(\mathbb{R}_+, dx)$. Let f_o be the odd extension of f to $\mathbb{R}$. If we expand the complex exponential in (32) using Euler's formula and recall that w is even, we see that, for any $j \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$\int_{\mathbb{R}} f_o \overline{\mathscr{F}h_{j,k}} = \frac{i}{2}\int_{\mathbb{R}} f_o g_{j,k} = i\int_0^\infty f g_{j,k}. \tag{35}$$

If $k \in \mathbb{Z}$ and $k \leq 0$, then $-k + 1 > 0$, so (35) and the relation $\mathscr{F}h_{j,k}(-x) = \mathscr{F}h_{j,-k+1}(x)$ imply that

$$\begin{aligned}
\int_{\mathbb{R}} f_o \overline{\mathscr{F}h_{j,k}} &= \int_{\mathbb{R}} f_o(-x)\overline{\mathscr{F}h_{j,k}(-x)}\,dx \\
&= -\int_{\mathbb{R}} f_o \overline{\mathscr{F}h_{j,-k+1}} = -i\int_0^\infty f g_{j,-k+1}.
\end{aligned}$$

Hence if f is orthogonal in $L^2(\mathbb{R}_+, dx)$ to every $g_{j,k}$ ($j \in \mathbb{Z}$, $k \in \mathbb{N}$), then f_o is orthogonal in $L^2(\mathbb{R})$ to every $\mathscr{F}h_{j,k}$ ($j, k \in \mathbb{Z}$). By the completeness noted above, $f_o = 0$, so $f = 0$. ∎

Proof of Theorem 3. By (9) and (34),

$$\hat{\psi}_{j,k}(\xi) = (2\pi)^{-1/2} 2^{-j/2+1} \xi^{-1/2} \sin(2^{-j}(k - 1/2)\xi) w(2^{-j}\xi), \quad (36)$$

or $\hat{\psi}_{j,k}(\xi) = \xi^{-1/2} g_{j,k}(\xi)$. Hence, by (10),

$$\langle \psi_{j,k}, \psi_{j',k'} \rangle = \langle \hat{\psi}_{j,k}, \hat{\psi}_{j',k'} \rangle_\xi = \langle g_{j,k}, g_{j',k'} \rangle.$$

So the orthonormality of $\{\psi_{j,k}\}$ in $L^2(\mathbb{R}_+, dx)$ follows from Lemma 4.

For completeness, suppose $f \in L^2(\mathbb{R}_+, dx)$ and $\langle f, \psi_{j,k} \rangle = 0$ for all $(j, k) \in \mathbb{Z} \times \mathbb{N}$. Since $\hat{}$ is unitary, $\sqrt{\xi}\hat{f}(\xi) \in L^2(\mathbb{R}_+, d\xi)$ and, for each $j \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$\left\langle \sqrt{\xi}\hat{f}(\xi), g_{j,k}(\xi) \right\rangle = \langle \hat{f}, \hat{\psi}_{j,k} \rangle_\xi = \langle f, \psi_{j,k} \rangle = 0.$$

By Lemma 4, $\sqrt{\xi}\hat{f}(\xi) = 0$ a.e. Hence $\hat{f} = 0$ and therefore $f = 0$ in $L^2(\mathbb{R}_+, dx)$. ∎

Observe that (16) follows from (30) and (36). We will use the following lemma to estimate $\psi_{j,k}$.

LEMMA 5. *For $h \in L^1(\mathbb{R}_+)$ and $x, b \in \mathbb{R}_+$, define*

$$S(h)(x, b) = \int_0^\infty h(\xi) \sin(b\xi) \sqrt{\xi} J_\nu(x\xi)\, d\xi$$

and

$$C(h)(x, b) = \int_0^\infty h(\xi) \cos(b\xi) \sqrt{\xi} J_\nu(x\xi)\, d\xi.$$

Suppose g: $\mathbb{R}_+ \to \mathbb{C}$ is C^2 and $\operatorname{supp} g \subseteq [c, d]$ for some $0 < c < d < \infty$. Then for $x \neq b$,

$$S(g)(x, b) = \frac{-2b}{x^2 - b^2} C(g')(x, b) + \frac{1}{x^2 - b^2} S(\tilde{L}g)(x, b) \quad (37)$$

and

$$C(g)(x, b) = \frac{2b}{x^2 - b^2} S(g')(x, b) + \frac{1}{x^2 - b^2} C(\tilde{L}g)(x, b), \quad (38)$$

where $\tilde{L}$ is as in (1).

Proof. Note by (6) that $\tilde{L}_{(\xi)}(\sqrt{\xi}J_\nu(x\xi)) = x^2\sqrt{\xi}J_\nu(x\xi)$, where $\tilde{L}_{(\xi)}$ denotes the operator in (1) acting in the ξ variable. Hence

$$\begin{aligned}(x^2 - b^2)S(g)(x,b) &= \int_0^\infty g(\xi)\sin(b\xi)\tilde{L}_{(\xi)}\big(\sqrt{\xi}J_\nu(x\xi)\big)\,d\xi - b^2S(g)(x,b)\\ &= \int_0^\infty \tilde{L}_{(\xi)}(g(\xi)\sin(b\xi))\sqrt{\xi}J_\nu(x\xi)\,d\xi - b^2S(g)(x,b),\end{aligned}$$

by integration by parts (the boundary terms vanish by the support conditions on g). But

$$\tilde{L}_{(\xi)}(g(\xi)\sin(b\xi)) = \sin(b\xi)\tilde{L}g(\xi) - 2bg'(\xi)\cos(b\xi) + b^2g(\xi)\sin(b\xi).$$

Substituting this yields (37).

We obtain (38) in the same way. ∎

Let χ_E denote the characteristic function of a set E.

THEOREM 6. *Suppose w in* (34) *belongs to $C^{2M}(\mathbb{R}_+)$ for some $M \in \mathbb{N}$. Then for $m = 0, 1, 2, \dots,$ there exists $c_{m,M} < +\infty$ such that for all $j \in \mathbb{Z}$, $k \in \mathbb{N}$, and $x > 0$,*

$$\left|\psi_{j,k}^{(m)}(x)\right| \le c_{m,M}2^{j(1/2+m)}(1 + 2^jx/k)^{-M}(1 + 2^j|x - 2^{-j}k|)^{-M} \times\left\{(2^jx)^{\nu-m+1/2}\chi_{(0,2^{-j})}(x) + \chi_{[2^{-j},\infty)}(x)\right\}.$$

Proof. By (18), which follows from (34), it is sufficient to prove

$$\left|\psi_{0,k}^{(m)}(x)\right| \le c_{m,M}(1 + x/k)^{-M}(1 + |x - k|)^{-M} \times\left\{x^{\nu-m+1/2}\chi_{(0,1)}(x) + \chi_{[1,\infty)}(x)\right\}$$

for all $k \in \mathbb{N}$ and $x > 0$. Let

$$\begin{aligned}G_{0,k}(x) &= x^{-1/2}\psi_{0,k}(x)\\ &= \sqrt{\frac{2}{\pi}}\int_{2\pi/3}^{8\pi/3}\sin((k - 1/2)\xi)w(\xi)J_\nu(x\xi)\sqrt{\xi}\,d\xi.\end{aligned}$$

For notational convenience, let

$$H_\ell(x) = x^\ell\chi_{(0,1)}(x) + x^{-1/2}\chi_{[1,\infty)}(x).$$

By the Leibniz rule, it is sufficient to prove that

$$\left|G_{0,k}^{(m)}(x)\right| \leq c_{m,M}(1+x/k)^{-M}(1+|x-k|)^{-M} H_{\nu-m}(x). \tag{39}$$

Note that

$$G_{0,k}^{(m)}(x) = \sqrt{\frac{2}{\pi}} \int_{2\pi/3}^{8\pi/3} \sin((k-1/2)\xi) w(\xi) \xi^m J_\nu^{(m)}(x\xi) \sqrt{\xi}\, d\xi, \tag{40}$$

by differentiating under the integral sign.

By (5), $|J_\nu^{(m)}(x)| \leq c_m x^{\nu-m}$ for $0 < x < 1$, for all $m \in \mathbb{N} \cup \{0\}$. Also $|J_\nu(x)| \leq cx^{-1/2}$ for $x > 1$ (see, e.g., [6, p. 139]; this holds for all $\nu \in \mathbb{R}$). By the relation $J_\nu'(x) = J_{\nu-1}(x) - \frac{\nu}{x} J_\nu(x)$ ([6, p. 133]; this also holds for all $\nu \in \mathbb{R}$) it follows that

$$\left|J_\nu^{(m)}(x)\right| \leq c_{\nu,m} H_{\nu-m}(x) \tag{41}$$

for all $x > 0$.

Suppose that $|x-k| \leq 1$. Then (39) reduces to $|G_{0,k}^{(m)}| \leq c_{m,M} H_{\nu-m}$, which follows from (40) and (41) by simple magnitude estimates.

Now suppose $|x-k| > 1$ and, to begin with, $m = 0$. By definition,

$$G_{0,k}(x) = \sqrt{\frac{2}{\pi}} S(w)(x, k-1/2),$$

where S is as in Lemma 5. By assumption, w is a C^{2M} function supported on $[2\pi/3, 8\pi/3]$. Note that applying the derivative or L does not increase the support and decreases the degree of smoothness by at most 2. We can apply (37) and (38) M times and write $G_{0,k}$ as a linear combination of terms of the form

$$\begin{gathered}\frac{(k-1/2)^\ell}{\left(x^2-(k-1/2)^2\right)^M} S(g)(x, k-1/2) \quad \text{or} \\ \frac{(k-1/2)^\ell}{\left(x^2-(k-1/2)^2\right)^M} C(g)(x, k-1/2),\end{gathered} \tag{42}$$

where $0 \leq \ell \leq M$ and g is some M-fold iteration of either the derivative or $\tilde{L}$ applied to w. Note that g is supported in $[2\pi/3, 8\pi/3]$ and is continuous (since $w \in C^{2M}$). Hence (41) and simple size estimates yield

$$\left|G_{0,k}(x)\right| \leq c_M (k-1/2)^M \left(x^2-(k-1/2)^2\right)^{-M} H_\nu(x),$$

which is equivalent to (39) for $|x-k| > 1$ and $m = 0$.

For $m > 0$, we take the mth derivative of the terms in (42). Let $m = m_1 + m_2 + m_3$, where each m_i is a nonnegative integer. By the Leibniz rule, we obtain terms which are constant multiples of

$$D^{(m_1)} \frac{(k - 1/2)^{\ell}}{(x + (k - 1/2))^M} D^{(m_2)} \frac{1}{(x - (k - 1/2))^M}$$
$$\times D^{(m_3)} \int_0^\infty \sin((k - 1/2)\xi) g(\xi) \xi^{m_3} J_\nu^{(m_3)}(x\xi)\sqrt{\xi}\, d\xi,$$

or similar terms with $\cos((k - 1/2)\xi)$ in place of $\sin((k - 1/2)\xi)$. For the derivative of order m_1, we obtain an estimate of the form $c_m(k - 1/2)^{\ell}(x + (k - 1/2))^{-M-m_1} \le c_m k^M (x + k)^{-M}$ (where the constant c_m is allowed to have different values at different occurences). For the m_2 derivative, we obtain a bound of the form $c_m|x - k|^{-M-m_2} \le c_m(1 + |x - k|)^{-M}$ in the same way, using the assumption that $|x - k| > 1$. For the integral term, (41) and the support conditions yield a bound of the form $c_M H_{\nu - m_3}(x) \le c_M H_{\nu - m}(x)$. Putting these estimates together gives (39). ∎

Note that (17) is the case $m = 0$ of Theorem 6.

COROLLARY 7. *For each $j \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have $\psi_{j,k} \in \mathcal{D}_L$.*

Proof. By Theorem 6, $\psi_{j,k} \in C^\infty(\mathbb{R}_+)$, hence $\psi_{j,k}, \psi'_{j,k} \in AC_{loc}(\mathbb{R}_+)$. For $0 < x < 1$, $|\psi_{j,k}(x)| \le c_j x^{\nu + 1/2}$ and $|\psi'_{j,k}(x)| \le c_j x^{\nu - 1/2}$, also by Theorem 6. These estimates guarantee that $\lim_{x \to 0+} \psi_{j,k}(x) = 0$ and that $\psi_{j,k}$ and $\psi'_{j,k}$ are square integrable on $(0, 1)$, where we use the assumption that $\nu > 0$ for the case of $\psi'_{j,k}$. Since we assume $w \in C^2$, Theorem 6 applies with $M = 1$ to show that $\psi_{j,k}$ and $\psi'_{j,k}$ have decay at ∞ of order at least x^{-2}. Hence $\psi_{j,k}, \psi'_{j,k} \in L^2(\mathbb{R}_+)$.

By (2), what remains is to show $\tilde{L}\psi_{j,k} \in L^2(\mathbb{R}_+)$. The estimates of Theorem 6 show that both terms $-\psi''_{j,k}$ and $(\nu^2 - 1/4)x^{-2}\psi_{j,k}$ in $\tilde{L}\psi_{j,k}$ are square integrable on $[1, \infty)$. To consider x near 0, first bring $\sqrt{x}$ and then $\tilde{L}$ in (34) inside the integral, and apply (6) to obtain

$$\tilde{L}\psi_{j,k}(x) = (2\pi)^{-1/2} 2^{-j/2+1}$$
$$\times \int_0^\infty \sin(2^{-j}(k - 1/2)\xi) w(2^{-j}\xi) \xi^2 \sqrt{x} J_\nu(x\xi)\sqrt{\xi}\, d\xi. \quad (43)$$

From this and the compact support of w, trivial estimates show that $\tilde{L}\psi_{j,k}$ is bounded on $(0, 1)$ (in fact, $\tilde{L}\psi_{j,k}(x) = O(x^{\nu + 1/2})$). This shows that $\tilde{L}\psi_{j,k}$ belongs to $L^2(\mathbb{R}_+)$ and completes the proof. ∎

Similar results hold for $\varphi_{j,k}$ also, but we do not state these since they will not be used.

4. NORM ESTIMATES FOR BESSEL WAVELETS

In this section we use the norm characterizations in Theorem 2 to derive analogous results for Bessel wavelets.

THEOREM 8. *Let $s \in \mathbb{R}$. For $f \in L^2(\mathbb{R}_+, dx)$,*

$$\|f\|_{H_B^s} \approx \left(\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}} \left| 2^{js} \langle f, \psi_{j,k} \rangle \right|^2 \right)^{1/2}.$$

Our primary interest is in the case $s = 1$ (i.e., (19)), but the general case is no more difficult to prove. We require the following lemma.

LEMMA 9. *Suppose $(j,k),(j',k') \in \mathbb{Z} \times \mathbb{N}$ and that w in (34) belongs to C^M for some integer $M \geq 2$.*

$$\text{If } j' - j \notin \{2,3,4,5\}, \text{ then } \langle \psi_{j,k}, \varphi_{j',k'} \rangle = 0. \tag{44}$$

If $j' - j \in \{2,3,4,5\}$, there exists a constant $c_M < \infty$, independent of j, j', k, and k', such that

$$\left| \langle \psi_{j,k}, \varphi_{j',k'} \rangle \right| \leq c_M \left(1 + \left| 2^{j'-j}(k - 1/2) - k'\pi \right| \right)^{-M}. \tag{45}$$

Proof. By (10), (24), and (36),

$$\begin{aligned}
\langle \psi_{j,k}, \varphi_{j',k'} \rangle &= \langle \hat{\psi}_{j,k}, \hat{\varphi}_{j',k'} \rangle_\xi \\
&= 2\pi^{-1/2} 2^{-(j+j')/2} \\
&\quad \times \int_0^\infty w(2^{-j}\xi) \gamma(2^{-j'}\xi) \sin(2^{-j}(k - 1/2)\xi) \\
&\quad \times \sin(2^{-j'} k'\pi\xi)\, d\xi \\
&= 2\pi^{-1/2} 2^{(j'-j)/2} \\
&\quad \times \int_0^\infty w(2^{j'-j}y) \gamma(y) \sin(2^{j'-j}(k + 1/2)y) \sin(k'\pi y)\, dy \\
&= \pi^{-1/2} 2^{(j'-j)/2} \int_0^\infty w(2^{j'-j}y) \gamma(y) \\
&\quad \times \{\cos[(k'\pi - 2^{j'-j}(k - 1/2))y] \\
&\qquad - \cos[(k'\pi + 2^{j'-j}(k - 1/2))y]\}\, dy.
\end{aligned}$$

By (21) and (30), $w(2^{j'-j}y)\gamma(y)$ is identically 0 unless $j'-j \in \{2,3,4,5\}$, which establishes (44). For the four remaining possibilities for $j'-j$, $w(2^{j'-j}y)\gamma(y)$ is a C^M compactly supported function. Integrating by parts M times in the last integral yields

$$\left|\langle \psi_{j,k}, \varphi_{j',k'}\rangle\right| \le c_M\left(1+\left|2^{j'-j}(k-1/2)-k'\pi\right|\right)^{-M} + c_M\left(1+\left|2^{j'-j}(k-1/2)+k'\pi\right|\right)^{-M}.$$

Since $k, k' \ge 1$, the first term dominates. ∎

The estimates in Lemma 9 control the "change of coordinates" matrix which allows us to pass from Theorem 2 to Theorem 8.

Proof of Theorem 8. For $(j,k) \in \mathbb{Z}\times\mathbb{N}$ let

$$a_{j,k} = 2^{js}\langle f, \psi_{j,k}\rangle \qquad \text{and} \qquad b_{j,k} = 2^{js}\langle f, \varphi_{j,k}\rangle.$$

Define vectors $a = (a_{j,k})_{(j,k)\in\mathbb{Z}\times\mathbb{N}}$ and $b = (b_{j,k})_{(j,k)\in\mathbb{Z}\times\mathbb{N}}$. Also define infinite matrices $R = (r_{j,k;j',k'})$ and $S = (s_{j,k;j',k'})$, where (j,k) and (j',k') run through $\mathbb{Z}\times\mathbb{N}$, by setting

$$r_{j,k;j',k'} = 2^{(j-j')s}\langle \psi_{j,k}, \varphi_{j',k'}\rangle \qquad \text{and} \qquad s_{j,k;j',k'} = 2^{(j-j')s}\langle \varphi_{j,k}, \psi_{j',k'}\rangle.$$

Applying (25) to $\psi_{j,k}$,

$$a_{j,k} = 2^{js}\left\langle f, \sum_{j'\in\mathbb{Z}}\sum_{k'\in\mathbb{N}}\left\langle \psi_{j,k}, \varphi_{j',k'}\right\rangle \varphi_{j',k'}\right\rangle = \sum_{j'\in\mathbb{Z}}\sum_{k'\in\mathbb{N}} r_{j,k;j'k'}b_{j',k'},$$

or $a = Rb$. Similarly, expanding $\varphi_{j,k}$ using Theorem 3, we obtain $b = Sa$.

Since we assume $w \in C^2$, (45) holds with $M = 2$. From this and (44), it follows that

$$\sup_{(j,k)\in\mathbb{Z}\times\mathbb{N}} \sum_{(j',k')\in\mathbb{Z}\times\mathbb{N}} |r_{j,k;j',k'}| < \infty$$

and

$$\sup_{(j',k')\in\mathbb{Z}\times\mathbb{N}} \sum_{(j,k)\in\mathbb{Z}\times\mathbb{N}} |r_{j,k;j',k'}| < \infty,$$

and similarly for S. By Schur's lemma (with weights equal to 1) (see e.g., [5, p. 185]), R and S act boundedly on $\ell^2(\mathbb{Z}\times\mathbb{N})$. Hence

$$\|a\|_{\ell^2(\mathbb{Z}\times\mathbb{N})} \approx \|b\|_{\ell^2(\mathbb{Z}\times\mathbb{N})} \approx \|f\|_{H^s_B},$$

by Theorem 2. ∎

5. THE GALERKIN SOLUTION OF $Lu = f$ USING BESSEL WAVELETS

We now consider the Galerkin approach to the numerical solution of the equation $Lu = f$ on $\mathbb{R}_+$, with $f \in \mathscr{R}(L)$ given. The solution is unique because $Lu = 0$ implies $\tilde{L}u = 0$, but any solution of $\tilde{L}u = 0$ is of the form $c_1 x^{\nu+1/2} + c_2 x^{-\nu+1/2}$, which does not belong to $L^2(\mathbb{R}_+)$ unless the constants c_1 and c_2 are both 0.

In the Galerkin approach, we consider a finite subset Λ of $\mathbb{Z} \times \mathbb{N}$. We set

$$S = \operatorname{span}\left\{\psi_{j,k} : (j,k) \in \Lambda\right\}.$$

We will approximate the solution u by an element of S. This is reasonable since $u \in L^2(\mathbb{R}_+)$ and $\{\psi_{j,k}\}_{(j,k) \in \mathbb{Z}\times\mathbb{N}}$ is an orthonormal basis for $L^2(\mathbb{R}_+)$. We write this approximation to u as

$$u_S = \sum_{(j,k)\in\Lambda} x_{j,k}\psi_{j,k}, \tag{46}$$

where the vector $x = (x_{j,k})_{(j,k)\in\Lambda}$ (whose components will depend on Λ, but we suppress this dependence notationally) is determined so that

$$\langle Lu_S, \psi_{j,k}\rangle = \langle f, \psi_{j,k}\rangle, \qquad \text{for all } (j,k) \in \Lambda. \tag{47}$$

Note that $u_S \in \mathscr{D}_L$ since $\psi_{j,k} \in \mathscr{D}_L$ for each j,k by Corollary 7. If we substitute (46) into (47), we obtain the linear system

$$\sum_{(j',k')\in\Lambda} \langle L\psi_{j',k'}, \psi_{j,k}\rangle x_{j',k'} = \langle f, \psi_{j,k}\rangle, \qquad \text{for all } (j,k) \in \Lambda. \tag{48}$$

We set $b = (b_{j,k})_{(j,k)\in\Lambda}$, where $b_{j,k} = \langle f, \psi_{j,k}\rangle$, and we define a matrix $A_S = (a_{j,k;j',k'})_{(j,k),(j',k')\in\Lambda}$ by setting $a_{j,k;j',k'} = \langle L\psi_{j',k'}, \psi_{j,k}\rangle$. Then (48) becomes $A_S x = b$.

Our goal is to show that we can precondition the linear system $Ax = b$ with a simple diagonal preconditioner, so that the result has a condition number bounded independent of the subspace S. Thus, using Bessel wavelets, we will obtain the same results for the Bessel operator that Jaffard ([8]) obtained using standard wavelets for uniformly elliptic Sturm–Liouville systems, as noted in the Introduction.

Define the diagonal matrix $D_S = (d_{j,k;j',k'})_{(j,k),(j',k')\in\Lambda}$ by

$$d_{j,k;j',k'} = \begin{cases} 2^j = 2^{j'} & \text{if } (j,k) = (j',k') \\ 0 & \text{otherwise.} \end{cases}$$

Then $A_S x = b$ is equivalent to

$$D_S^{-1} A_S D_S^{-1} D_S x = D_S^{-1} b. \tag{49}$$

Let $M_S = D_S^{-1} A_S D_S^{-1}$. Note that the components of M_S are

$$m_{j,k;j',k'} = 2^{-j-j'} \langle L\psi_{j',k'}, \psi_{j,k} \rangle. \tag{50}$$

Also define vectors $y = D_S x$ and $z = D_S^{-1} b$. Then (49) becomes

$$M_S y = z. \tag{51}$$

Our main point is that (51) is well-conditioned. Let C_1 and C_2 be the best constants in Theorem 8 for $s = 1$, i.e., the best values such that

$$C_1 \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}} \left| 2^j \langle f, \psi_{j,k} \rangle \right|^2 \le \|f\|_{H_B^1}^2 \le C_2 \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}} \left| 2^j \langle f, \psi_{j,k} \rangle \right|^2$$

for all $f \in H_B^1$. We denote the condition number of a matrix M by $c_\#(M)$.

THEOREM 10. *For any* $\Lambda \subseteq \mathbb{Z} \times \mathbb{N}$,

$$\|M_S\| \le C_2 \qquad \text{and} \qquad \|M_S^{-1}\| \le C_1^{-1}.$$

In particular, $c_\#(M_S) \le C_2 / C_1$.

Proof. Let $v = (v_{j,k})_{(j,k) \in \Lambda}$ be a vector of length 1. Let $w = D_S^{-1} v$; i.e., w has components $w_{j,k} = 2^{-j} v_{j,k}$. Define a function g by

$$g = \sum_{(j,k) \in \Lambda} w_{j,k} \psi_{j,k}.$$

Note $g \in \mathscr{D}_L$, by Corollary 7. Then by (50) and linearity,

$$\langle M_S v, v \rangle = \sum_{(j,k) \in \Lambda} \sum_{(j',k') \in \Lambda} \langle L\psi_{j',k'}, \psi_{j,k} \rangle 2^{-j'} v_{j',k'} 2^{-j} \overline{v_{j,k}} = \langle Lg, g \rangle. \tag{52}$$

Hence by (52) and (13), using Theorem 8 and the uniqueness of the Bessel wavelet transform, we have

$$\langle M_S v, v \rangle = \|g\|_{H_B^1}^2 \approx \sum_{(j,k) \in \Lambda} |2^j w_{j,k}|^2 = \sum_{(j,k) \in \Lambda} |v_{j,k}|^2 = 1.$$

More precisely, using the definitions of C_1 and C_2, we obtain

$$C_1 \le \langle M_S v, v \rangle \le C_2 \tag{53}$$

whenever $\|v\| = 1$. By (50), M_S is real and symmetric (since L is self-adjoint), and (53) shows that M_S is positive. Thus $\|M_S\|$ is the maximum eigenvalue of M_S, which by (53) is $\leq C_2$. Also, $\|M_S^{-1}\|$ is the reciprocal of the minimum eigenvalue of M_S. So, by (53), $\|M_S^{-1}\| \leq C_1^{-1}$. ∎

Let T be an operator of the form (20), with $0 < c_1 \leq a(x), b(x) \leq c_2 < \infty$ for all $x > 0$, for some constants c_1, c_2. We apply the Galerkin procedure to the equation $Tu = f$ using Bessel wavelets for L_ν for $\nu > 1/2$. We end up with the linear system

$$B_S x = y,$$

where B_S is the matrix with entries

$$b_{j,k;j',k'} = 2^{-j-j'}\langle T\psi_{j',k'}, \psi_{j,k}\rangle.$$

This system is also well-conditioned.

LEMMA 11. *The norms of B_S and B_S^{-1}, and hence $c_\#(B_S)$, are bounded with bounds independent of Λ.*

Proof. Let v, w, and g be as in the proof of Theorem 10. Following in the same way, we end up with

$$\langle B_S v, v\rangle = \langle Tg, g\rangle.$$

But

$$\langle Tg, g\rangle = \left\langle -(ag')' + bg/x^2, g\right\rangle = \langle ag', g'\rangle + \langle bg/x^2, g\rangle,$$

by integration by parts (justified by the estimates for $\psi_{j,k}$ in Theorem 6). But since $\nu > 1/2$, there exist positive constants c_3 and c_4 such that $c_3(\nu^2 - 1/4) \leq b(x) \leq c_4(\nu^2 - 1/4)$ for all $x \in \mathbb{R}_+$. Thus

$$\langle Tg, g\rangle \approx \langle g', g'\rangle + \left\langle \left(\nu^2 - 1/4\right)g/x^2, g\right\rangle = \langle Lg, g\rangle,$$

by integrating by parts again and noting that each term is nonnegative (since $\nu > 1/2$). Then the proof runs as before. ∎

In addition to having a bounded condition number, the matrix M_S in (51) has a second key computational property: it is sparse.

LEMMA 12. (i) *The entries of M_S satisfy*

$$m_{j,k;j',k'} = 0 \qquad \text{if } |j' - j| > 1. \tag{54}$$

(ii) *Suppose w in (34) belongs to C^M for some integer $M \geq 2$. There exists a positive constant C_M independent of j, k, j', k', and Λ such that*

$$|m_{j,k;j',k'}| \leq c_M\left(1 + \left|2^{j-j'}(k' - 1/2) - (k - 1/2)\right|\right)^{-M} \qquad \text{if } |j' - j| \leq 1. \tag{55}$$

Proof. By (50), (10), (11), and (36),

$$\begin{aligned} m_{j,k;j',k'} &= 2^{-j-j'}\langle L\psi_{j',k'}, \psi_{j,k}\rangle \\ &= 2^{-j-j'}\left\langle \left(L\psi_{j',k'}\right)^{\wedge}, \hat{\psi}_{j,k}\right\rangle_{\xi} \\ &= 2\pi^{-1}2^{-3(j+j')/2}\int_0^{\infty} \xi^2 w(2^{-j}\xi)w(2^{-j'}\xi)\sin\left(2^{-j}(k-1/2)\xi\right) \\ &\qquad \times \sin\left(2^{-j'}(k'-1/2)\xi\right) d\xi \\ &= 2\pi^{-1}2^{3(j'-j)/2}\int_0^{\infty} y^2 w(2^{j'-j}y)w(y)\sin\left(2^{j'-j}(k-1/2)y\right) \\ &\qquad \times \sin((k'-1/2)y)\, dy. \end{aligned} \tag{56}$$

Now (54) follows from (30), and (55) follows as in the proof of Lemma 9. ▮

Note that this proof uses the relation (11), which is not available when we replace L with T as in (20). Nevertheless a certain decay in B_S (which gives a degree of sparseness after thresholding the entries) can be obtained from the localization estimates in Theorem 6.

6. NUMERICAL WORK

We do not have an explicit estimate of the constant C_2/C_1 in Theorem 10 bounding the condition number of each M_S. Hence in principle it could be huge. In this section we present numerical evidence which shows that it appears to be at most on the order of 4. In the process, certain implementation issues are addressed, such as finding a convenient exhausting sequence of wavelet-Galerkin subspaces.

We first discuss the computation of the matrix entries $m_{j,k;j',k'}$. We use (56), which shows that all we require for computation is the function w. We define w by (31), where ρ is given on $0 \le x \le 1$ by

$$\rho(x) = x^4(35 - 84x + 70x^2 - 20x^3),$$

as in [3, p. 119]. Then $\rho \in C^3(\mathbb{R})$ (note that $\rho'(x) = 140x^3(1-x)^3$ for $0 \le x \le 1$). By (54), $m_{j,k;j',k'} = 0$ if $|j'-j| > 1$. If $j' = j$, (56) shows that $m_{j,k;j',k'}$ depends only on k and k', so we set

$$\begin{aligned} R_0(k,k') &= m_{j,k;j,k'} \\ &= 2\pi^{-1}\int_{2\pi/3}^{8\pi/3} y^2 w^2(y)\sin((k-1/2)y)\sin((k'-1/2)y)\, dy. \end{aligned}$$

Note that R_0 is symmetric: $R_0(k,k') = R_0(k',k)$. If $j'-j=1$, (56) shows that $m_{j,k;j',k'}$ depends only on k and k', so we define

$$R_1(k,k') = m_{j,k;j+1,k'} = 2^{5/2}\pi^{-1}\int_{2\pi/3}^{4\pi/3} y^2 w(2y)w(y)\sin(2(k-1/2)y) \times \sin((k'-1/2)y)\,dy.$$

If $j'-j=-1$, then by the symmetry of M_S and the independence of j and j' in the previous case,

$$m_{j,k;j-1,k'} = m_{j-1,k';j,k} = R_1(k',k).$$

For $n \in \mathbb{Z}$, we define

$$H(n) = \pi^{-1}\int_{2\pi/3}^{8\pi 3} y^2 w^2(y)\cos(ny)\,dy \tag{57}$$

and

$$I(n) = 2^{3/2}\pi^{-1}\int_{2\pi/3}^{4\pi/3} y^2 w(2y)w(y)\cos((n-1/2)y)\,dy. \tag{58}$$

Then by trigonometry,

$$R_0(k,k') = H(k-k') - H(k+k'-1)$$

and

$$R_1(k,k') = I(2k-k') - I(2k+k'-1).$$

Thus we come down to computing the Fourier integrals $H(n)$ and $I(n)$. This must be done carefully. If we use a standard quadrature approximation to the integral, the highly oscillatory nature of the terms $\cos ny$ and $\cos((n-1/2)y)$ will lead to huge errors for n large, unless the number of terms in the sum is prohibitively great. Instead we follow the procedure described in [12, pp. 577–584], in particular using [12, (13.9.13)].

To verify that the condition number of M_S is bounded independent of S, we select a sequence of subspaces $S_\ell = \{\psi_{j,k}\colon (j,k)\in\Lambda_\ell\}$ such that $\{\Lambda_\ell\}_{\ell=1}^\infty$ is an increasing sequence of subsets of $\mathbb{Z}\times\mathbb{N}$ with $\bigcup_{\ell=1}^\infty \Lambda_\ell = \mathbb{Z}\times\mathbb{N}$. We begin by letting $\Lambda_1 = \{(0,1)\}$ and $\Lambda_2 = \{(0,1),(1,1),(1,2)\}$. If we continue with $\tilde{\Lambda}_3 = \{(0,1),(1,1),(1,2),(2,1),(2,2),(2,3),(2,4)\}$ and so on, then we will never include values of j that are less than 0, and we will not include additional values of k for each j. Instead, we obtain Λ_3 by shifting the first component of each entry of $\tilde{\Lambda}_3$ by subtracting 1. We obtain

$$\Lambda_3 = \{(-1,1),(0,1),(0,2),(1,1),(1,2),(1,3),(1,4)\}.$$

Actually, the matrices corresponding to $\tilde{\Lambda}_3$ and Λ_3 are the same, since the nonzero entries only depend on whether the j indices are the same or different by 1, as noted above. We define Λ_4 by expanding Λ_3 with pairs corresponding to $j = 2$,

$$\Lambda_4 = \Lambda_3 \cup \{(2,1),(2,2),(2,3),\dots,(2,8)\}.$$

We continue in this way. That is, to define Λ_ℓ when ℓ is even, we add $2^{\ell-1}$ terms at the next j level, starting at $k = 1$. For odd values of ℓ, we first shift all pairs of the previous set $\Lambda_{\ell-1}$ by subtracting 1 in the first component, and then we add $2^{\ell-1}$ pairs at the next level in the same way. More precisely, we have, for $\mu = 1,2,\dots,$

$$\Lambda_{2\mu} = \{(-\mu+1,1),(-\mu+2,1),(-\mu+2,2),(-\mu+3,1),\dots,\\ (\mu,1),(\mu,2),\dots,(\mu,2^{2\mu-1})\}$$

and, for $\mu = 0,1,2,\dots,$

$$\Lambda_{2\mu+1} = \{(-\mu,1),(-\mu+1,1),(-\mu+1,2),(-\mu+2,1),\dots,\\ (\mu,1),(\mu,2),\dots,(\mu,2^{2\mu})\}.$$

The sets Λ_ℓ are increasing and their union is $\mathbb{Z} \times \mathbb{N}$. Note that the cardinality of Λ_ℓ is $2^\ell - 1$.

We define a matrix $M_\ell = M_{S_\ell}$ corresponding to the set Λ_ℓ and the subspace S_ℓ. Its entries are the terms $m_{j,k;j',k'}$ corresponding to all (j,k) and (j',k') in Λ_ℓ. In particular, M_ℓ is a $(2^\ell - 1) \times (2^\ell - 1)$ matrix. For example, because the third and sixth elements of Λ_3 above are $(0,2)$ and $(1,3)$, the entry in the third row and sixth column (also the sixth row and third column) of M_3 is $m_{0,2;1,3} = R_1(2,3)$. We let the indices j,k in $m_{j,k;j',k'}$ label the row and j',k' label the column.

If the ordering of the pairs (j,k) for the rows and columns of M_ℓ is the same as in the listings above, as suggested by the preceding example, then M_ℓ has a simple block structure. There are ℓ square blocks down the diagonal, starting with a 1×1 block in the upper left corner (corresponding to the lowest j, which always has only $k = 1$). Going down the diagonal (left to right), this is followed by a 2×2 block corresponding to the second lowest level, then a 3×3 block, and so on, down to a $2^{\ell-1} \times 2^{\ell-1}$ block in the lower right corner, corresponding to the highest j level. All entries in these diagonal blocks are of the form $m_{j,k;j',k'}$ with $j = j'$; that is, these entries are of the form $R_0(k, k')$. In a block of size $2^p \times 2^p$, k and k' run from 1 to 2^p. Thus each of these diagonal blocks are submatrices of the largest one, and hence only the largest one needs to be computed to determine all of the diagonal blocks.

The only other nonzero terms (besides those in the diagonal blocks) are in rectangular blocks directly above and below the diagonal blocks, by (54). Since M_ℓ is symmetric, we describe only the blocks above the diagonal blocks. Above each $2^p \times 2^p$ block except for the case $p = 1$ (obviousiy nothing is above this entry in the first row), there is a $2^{p-1} \times 2^p$ block corresponding to terms of the form $m_{j,k;j',k'}$ with $j' = j + 1$, that is, to $R_1(k, k')$. In this block, k runs from 1 to 2^{p-1} and k' runs from 1 to 2^p. Thus again the smaller rectangular blocks are submatrices of the largest one.

To pass from $M_{\ell-1}$ to M_ℓ, one needs only to add three rectangular block matrices to $M_{\ell-1}$. The first is a $2^{\ell-1} \times 2^{\ell-1}$ block along the diagonal in the lower right corner consisting of the matrix $[R_0(k, k')]_{k,k'=1}^{2^{\ell-1}}$. The second is a $2^{\ell-2} \times 2^{\ell-1}$ block $[R_1(k, k')]_{1 \le k \le 2^{\ell-2}, 1 \le k' \le 2^{\ell-1}}$ above the new diagonal block. The third is the transpose of the second, added to the left of the new diagonal block.

By the fact that the entries $m_{j,k;j',k'}$ depend only on $j - j'$, the matrix M_ℓ can be regarded as corresponding to any ℓ levels, not necessarily those specified above. In practice, one would select the set Λ based on the region one is considering and other factors specific to a given application. However, since any finite set Λ is a subset of some Λ_ℓ, to confirm Theorem 10 it suffices to consider the sequence chosen above.

We obtained numerically the following table of values of the condition number of M_ℓ.

ℓ	$c_\#(M_\ell)$
2	1.9885
3	2.9350
4	3.6523
5	3.9938
6	4.1092
7	4.1609
8	4.1833
9	4.1921
10	4.1961

This confirms our main claim that the condition numbers are bounded and shows that they are sufficiently small for this method to be practical.

7. THE CASE $\nu = 0$

The case $\nu = 0$ presents some additional difficulties. In this case, the domain of $L = L_0$ is

$$\mathscr{D}_0 = \left\{ g \colon g, g' \in AC_{loc}(\mathbb{R}_+);\ \lim_{x \to 0+} g(x)/\left(\sqrt{x}\ \ln x\right) = 0;\ g, g', \tilde{L}g \in L^2 \right\} \tag{59}$$

([2, Theorem 3.1]). The general considerations of (4)–(13) still hold. Defining $\varphi_{j,k}$ as in (23), Theorems 1 and 2 hold, by the same arguments. Similarly, defining $\psi_{j,k}$ by (34), Theorem 3 is established in exactly the same way. The first time that the assumption $\nu > 0$ is used is in the proof of Corollary 7. Lemma 5 still holds for $\nu = 0$, and the estimates in Theorem 6 are also correct, by the same argument. It follows that $\psi_{j,k}, \psi_{j,k} \in AC_{loc}(\mathbb{R}_+)$. By Theorem 6, $\psi_{j,k}$ is dominated by a multiple of $x^{1/2}$ near 0, so $\lim_{x \to 0+} \psi_{j,k}(x)/(\sqrt{x}\ \ln x) = 0$. The estimates in Theorem 6 imply that $\psi_{j,k} \in L^2(\mathbb{R}_+)$ and that $\psi'_{j,k}$ is square integrable on $[1, \infty)$. But the estimates in Theorem 6 only give $|\psi'_{j,k}(x)| \le c_{j,k} x^{-1/2}$ for x near 0, so we cannot conclude that $\psi'_{j,k} \in L^2(\mathbb{R}_+)$. In fact, by (5), $J_0(x) = 1 + xh(x)$, with $h \in C^\infty(\mathbb{R})$. Substituting this in (34) and differentiating under the integral sign shows that $\psi'_{j,k}(x)$ is of the form $c_{j,k} x^{-1/2}$ plus a term which is bounded near 0. Curiously, however, the argument using (43) in the proof of Corollary 7 that shows that $\tilde{L}\psi_{j,k} \in L^2(\mathbb{R}_+)$ still applies. What happens is that there is a cancellation of singularities near the origin in the two terms of $\tilde{L}\psi_{j,k}$. Thus the only reason we cannot conclude that $\psi_{j,k} \in \mathscr{D}_0$ is the failure of $\psi'_{j,k}$ to be square integrable near the origin.

This causes some serious problems. For example, the natural equivalence

$$\langle \tilde{L}g, g \rangle = \langle -g'' - g/4x^2, g \rangle = \langle g', g' \rangle - \langle g/4x^2, g \rangle, \tag{60}$$

which holds for $g \in C_0^\infty(\mathbb{R}_+)$ by integration by parts, does not hold with $g = \psi_{j,k}$ because the right side is not defined. Similarly, in the Galerkin procedure, it is not reasonable to approximate the solution u of $L_0 u = f$ by $u_S = \sum_{(j,k) \in \Lambda} x_{j,k} \psi_{j,k}$ for $\psi_{j,k} \notin \mathscr{D}_0$. Hence the Friedrichs extension is not the natural extension of $\tilde{L}|_{C_0^\infty}$ for the Bessel–Galerkin procedure. Instead we work with the maximal extension $L_{\max}$, which is the restriction of $\tilde{L}$ to its maximal domain

$$\mathscr{D}_{\max} = \left\{ g \colon g, g' \in AC_{loc}(\mathbb{R}_+);\ g \in L^2(\mathbb{R}_+);\ \tilde{L}g \in L^2(\mathbb{R}_+) \right\}.$$

From what we have noted, $\psi_{j,k} \in \mathscr{D}_{\max}$ for every j, k. The disadvantage of $L_{\max}$ is that it is not a self-adjoint operator. In particular, we cannot conclude that (11) and (13) hold for all $g \in \mathscr{D}_{\max}$.

However, from (43) and (9),

$$\begin{aligned}(L_{\max}\psi_{j,k})\hat{}(\xi) &= (2\pi)^{-1/2}2^{-j/2+1}\xi^{3/2}\sin(2^{-j}(k-1/2)\xi)w(2^{-j}\xi)\\ &= \xi^2\hat{\psi}_{j,k}(\xi), \end{aligned} \tag{61}$$

for each j, k. Thus, although we cannot use (11), which follows from the spectral theorem applied to the self-adjoint operator L_0, we still obtain the conclusion of (11) for $L_{\max}$ and $g = \psi_{j,k}$ by a direct argument. Note that (61) implies that

$$\langle L_{\max}\psi_{j,k}, \psi_{j',k'}\rangle = \langle \psi_{j,k}, L_{\max}\psi_{j',k'}\rangle \tag{62}$$

for all j, k, j', k' by (10) since

$$\begin{aligned}\left\langle (L_{\max}\psi_{j,k})\hat{}, \hat{\psi}_{j',k'}\right\rangle_\xi &= \left\langle \xi^2\hat{\psi}_{j,k}(\xi), \hat{\psi}_{j',k'}(\xi)\right\rangle_\xi\\ &= \left\langle \hat{\psi}_{j,k}(\xi), \xi^2\hat{\psi}_{j',k'}(\xi)\right\rangle_\xi = \left\langle \hat{\psi}_{j,k}, (L_{\max}\psi_{j',k'})\hat{}\right\rangle_\xi.\end{aligned}$$

Thus $L_{\max}$ restricted to a Bessel–Galerkin subspace $S = \operatorname{span}\{\psi_{j,k}: (j,k) \in \Lambda\}$ (where Λ is a finite set) is self-adjoint. This turns out to be all that is needed to obtain the primary results for $L_{\max}$.

First, Theorem 8 in Section 4 carries through with no change because its proof does not involve the operator L. Now consider the equation $L_{\max}u = f$, with $f \in \mathscr{R}(L_{\max})$, the range of $L_{\max}$. The solution u is unique, because a solution of $\tilde{L}u = 0$ is of the form $c_1x^{1/2} + c_2x^{1/2}\ln x$, which is not in $L^2(\mathbb{R}_+)$ unless it is identically 0. Let S be a Bessel–Galerkin subspace, as above, and suppose $u_S = \sum_{(j,k)\in\Lambda}x_{j,k}\psi_{j,k}$. Then (47) with L replaced with $L_{\max}$ makes sense because $S \subseteq \mathscr{D}_{\max}$. Thus we obtain the linear system (48) with L replaced with $L_{\max}$. We precondition as in Section 5, obtaining the matrix equation (51), where M_S has components defined in (50) with $L_{\max}$ in place of L. Note that M_S is symmetric by (62). Theorem 10, which states the boundedness of the condition number of M_S, still holds. There are two ways to see this. First, one can note that the only step in the proof that requires modification is the use of (13) for $g = \sum_{(j,k)\in\Lambda}w_{j,k}\psi_{j,k} \in S$. Previously we had (13) automatically, for all elements in the domain of the operator. This is not so clear here, but (11) still holds for $g \in S$ by (61), which implies (13). However, the second way to see Theorem 10 is easier: just note (e.g., by (56)) that M_S is independent of ν and use the result for $\nu > 0$. Similarly, we obtain Lemma 12.

Of course there is no analog of Lemma 11 because it requires $\nu > 1/2$. The approach just outlined works also for $\nu > 0$. This may be useful for considering the equation $Lu = f$ for $f \in \mathscr{R}(L_{\max}) \setminus \mathscr{R}(L)$, where here $L_{\max}$ is the maximally defined operator for $\nu > 0$. However, it is more natural to work with L, when possible, and it is convenient to know that $\psi_{j,k} \in \mathscr{D}_L$ and that equations like (60) with $g = \psi_{j,k}$ are correct for $\nu > 0$.

8. CONCLUSION

Much of the approach we have taken can be generalized to a certain class of Sturm–Liouville operators. In this case, there is a transformation $\hat{}$ with inverse $\check{}$ defined as in (7) and (9), but with $\sqrt{x}J_\nu(x\xi)$ replaced with a kernel of the more general form $K(x, \xi)$ and with $\xi\, d\xi$ replaced with a spectral measure $d\rho(\xi)$ on $\mathbb{R}$. This Sturm–Liouville transform is a unitary map from $L^2(\mathbb{R}_+, dx)$ to $L^2(\mathbb{R}, d\rho)$ which diagonalizes the Sturm–Liouville operator L in the sense that (11) holds (by the spectral theorem for self-adjoint differential operators, as in [1, p. 192]). If we assume that $d\rho$ is supported on $\mathbb{R}_+$, is absolutely continuous and is positive and bounded away from 0, then many of the results of this paper carry over, such as Theorems 1, 2, 3, and 8. However, the estimates in Theorem 6 are not certain in general. For the Bessel case presented here, these are simplified by the explicit nature of the kernel $K(x, \xi) = \sqrt{x}J_\nu(x\xi)$.

When discussing the Galerkin procedure for numerically solving $Lu = f$, we assume that the wavelet coefficients $\langle f, \psi_{j,k} \rangle$ are given. We have not discussed how to compute these coefficients efficiently. In the general Sturm–Liouville setting, under certain estimates on the "father wavelet," there is a fast algorithm which is similar to the usual wavelet algorithm. Moreover, this leads to a fast algorithm for computing the Sturm–Liouville transform itself. However, at this point it is not clear how restrictive these conditions on the father wavelet are. These topics are discussed in the setting of more general Sturm–Liouville operators in [13]. We hope to address these points in subsequent papers.

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